Work performed by a Classical-“reversible”-Carnot cycle: Raising’s distribution for the small ”driving weights”.

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Abstract

The expansions or the compressions of the ideal gas in the quasi-static Carnot cycle, can be performed (on adiabatic or isothermal way) by slowly increasing or decreasing the external pressure by means of small weights acting on the piston of the vessel containing the gas. We call them shortly the “driving weights” (dw). Let N be their number, a large one.

To determine the work performed by the ideal gas in the cycle the “driving weights” must be handled carefully. If we let them move on-off the piston only horizontally, their vertical motions will be due only to the gas. Here we show that, at the end, while some of them will have moved down (will have negative raising) the remaining ones (the majority) will have moved up (will have positive raising) so that the total work performed by the ideal gas equals the total variation of the gravitational potential energy of the “driving weghts”.

The cycle is performed in $2N$ time-steps. For each step t_i, $i \in (1,..,2N)$, we give $H(t_i)$ and $\Delta H(t_{i-1}, t_i)$, respectively the height and the raising of the piston. Moreover the overall raising of the individual $dw\prime s$ (i.e. h_k, $k \in (1...N)$), and their distribution are given in simple, general cases. The efficiency and the dissipated work are also evaluated.

This paper is aimed at imparting a deeper understanding of the ideal Carnot Engine and may also be useful as a segment in a didactic path on elementary calculus and statistics.

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1 Introduction

To perform an ideal gas quasi-static-"reversible" [1]-Carnot-cycle, we need an heat source, an heat sink, a vessel with a mobile piston and many small "driving weights" to increase or decrease slowly the external pressure (both for the isothermal and adiabatic processes).[2]-[5]

To determine the work performed by the ideal gas in the cycle the "driving weights" (dw) must be handled carefully. Therefore we let them move on-off the piston only horizontally. To this end we assume that the handle of the piston be endowed of so many consoles that we can move each dw horizontally from (or to) the corresponding fixed console as reported in Figure 1.

The piston is mobile, so when we increase the external pressure the last coming dw (together with the previous ones) goes down. In the opposite case (expansion) the chosen dw goes at rest on the console in front of it and the others go up.

"In this way you can find out the work performed by the ideal gas in the cycle: you will find it in the increased potential energy of some of the $dw's$", I said in the last 25 years in the thermodynamic class at the University of Naples. Sometimes in these years some clever student replayed: "I have thought a lot about that and it seems to me that the $dw's$ go down". My reply has been always "You have to think more, because the work performed by the gas must necessarily increase the potential Energy of the $dw's$ ".[6]

Recently I decided to do a Java Applet [7] to show the raising of the $dw's$ and, in writing down some elementary relations, I have found out that the clever students were not totally wrong. In the following I report on a simple Carnot-cycle done through the ideal device given in Figure 1 and show that, at the end of the cycle, some of the $dw's$ will have moved down (will have negative raising) while the remaining ones (the majority) will have moved up (will have positive raising) so that the total work performed by the ideal gas equals the total variation of the gravitational potential energy of the "driving weights". In particular in Sec.2 is given an example of a Carnot-cycle done in $2N$ steps. In the first N steps the $dw's$ are added on the piston to perform first an isothermal compression and then an adiabatic compression. In the remaining N steps the $dw's$ are removed from the piston in order to return to the initial state. The cycle is reported in Figure 2. The height and the raising of the piston et each step i.e. $H(t_i)$ and $\Delta H(t_{i-1}, t_i) = H(t_i) - H(t_{i-1})$, are also calculated and reported in Figure 3 and Figure 4

In Sec.3 we evaluate the raising of each dw at the end of the cycle. To find the raising of the dw which was put on the piston at one given time step, one must know when it has been removed. The order in which the $dw's$ are removed is quite arbitrary and actually there are $N!$ ways in which the removing process

Figure 1: a) The adiabatic vessel with some $dw's$ on the piston; $a = 10\ cm$, $b = 27\ cm$, $c = 54\ cm$. $V_{box} = abc = 14580cm^3 = 14.58\ l$. $V_0 = 22.4l = V_{box} + H_0 S$, where $H_0 = 78.2\ cm$, $S = aa = 100\ cm^2$ b) The adiabatic vessel together with two supports for the $dw's$

can be done. We analyze two main non-random removing processes: for both of them the overall raising of the single dw is related to the sum of the raisings of the piston for the time-steps in which the dw 'lived' on the piston and hence to the difference of the height of the piston between the time at which it was removed and at which it was added on the piston . The results are shown in Figure 6, Figure 7 and Figure 8. In Figure 5 one can understand, directly from the $P-V$ representation of the cycle, why some $dw's$ have negative raising. In Sect. 4 the relation which connects the raising of the single dw to the expansion of the gas is obtained in an alternative way. The efficiency of the engine is evaluated in Sec.5 while the entropic gains are estimated in Appendix.

2 The cycle and the raising of the piston

One mole of ideal gas (e.g. dry air) is contained in the vessel shown in Figure 1, and it is initially in thermal contact with the heat reservoir at $T_0 = 273.15\ K$. The initial volume of the gas is $V_0 = 22.4\ \ l$. Let $P_0 = 1\ atm = 101.3\ KPa$ be the external pressure on the piston and let the piston be massless so that at the initial stage also the pressure of the gas is P_0. For the dry air $\gamma = \frac{C_p}{Cv} = 1.4$.

The cycle we realize is reported in Figure 2. It is done in $2N$ time-steps. In the first N steps the pressure $P(t_i)$, $i \in (1..N)$ increases. At each step we add on the piston a single dw. Let its mass be $m = 0.1\ Kg$. Since the surface of the piston is $S\ = 100\ cm^2$, this implies that at each step we are increasing the pressure of a relatively small amount. The degree of irreversibility for this step-wise process is evaluated in Appendix.

In the first N steps we have

$$\frac{P(t_i)}{P_0} = \frac{z+i}{z}$$

where $P(t_0) \equiv P_0$ is the initial pressure, i is the number of $dw's$ on the piston at step t_i and $z = 1033$ is the number of grams whose weight on 1 cm^2 give the pressure of 1 atm. Therefore $P(t_i) = P_0 + i\Delta P$ with $\Delta P = \frac{1}{z}P_0 = \frac{1}{1033}P_0$. In the second N steps the $dw's$ are removed. As at each step a single dw is removed, for $P(t_{N+l})$, $l \in (1...N)$ we have $P(t_{N+l}) = P_0 + (N-l)\Delta P$ and

$$\frac{P(t_{N+l})}{P_0} = \frac{z+N-l}{z} = \frac{P(t_{N-l})}{P_0}$$

but obviously $V(t_{N+l}) \neq V(t_{N-l})$.

The choice of the values of N and n_1 (the number of steps of the isothermal process at T_0) are somehow free, even if they depend on the geometry of the vessel and on choice of the mass of the single dw. Here we find convenient the values $N = 610$ and $n_1 = 410$.

Figure 2: The Carnot-cycle. It is step-wise but the steps are very small. $P_0V_0- > P_AV_A$ is the isothermal process at $T_0 = 273.15$, ($P_A = 1.3969$ atm, $V_A = 16.034\ l$). $P_AV_A- > P_BV_B$ is an adiabatic process ($P_B = 1.5905$ atm and $V_B = 14.614\ l$). $P_BV_B- > P_CV_C$ is the isothermal process at $T = 283.47$ ($P_C = 1.1385\ atm\ and\ V_C = 20.416l$). $P_CV_C- > P_0V_0$ is the final adiabatic process.

The state $P_A V_A$ is therefore reached in n_1 time-steps in isothermal conditions. The height of the piston at each step, i.e. $H(t_i)$ can be determined from the ideal gas state equation $P(t_i)V(t_i) = RT_0$, which, together with the relation

$$V(t_i) = V_{box} + H(t_i)S \tag{1}$$

gives

$$H(t_i) = H_0 + \frac{V_0}{S}(\frac{P_0}{P(t_i)} - 1) \tag{2}$$

From this we can evaluate step by step the raising of the piston $\Delta H(t_{i-1}, t_i) = H(t_i) - H(t_{i-1}) \equiv \Delta H(t_i)$, therefore

$$\Delta H(t_i) = \frac{V_0 P_0}{S}\left(\frac{1}{P(t_i)} - \frac{1}{P(t_{i-1})}\right) \tag{3}$$

Let denote with $H_1(t_i)$ and $\Delta H_1(t_i)$ the values given by relations (2) and (3), which are relative to the first n_1 steps.

It is obvious that for each isothermal step $\Delta T(t_i) = 0$; not obvious is the estimate of the Entropic change of the Universe for an isothermal step, in the Appendix we find $\Delta \mathcal{S}_U(t_i) \simeq R(\frac{\Delta P}{P})^2$, where R is the Universal gas constant.

The state $P_B V_B$ is reached in $n_2 = 200$ time-steps in adiabatic conditions (thermal contact replaced by an adiabatic wall). For the adiabatic steps we cannot use $P(t_i)V(t_i)^\gamma = Const$. Nevertheless by means of the First Law of thermodynamics and the equation of state of the ideal gases in the Appendix we find that

$$\Delta V(t_i) = -\frac{V(t_{i-1})}{\gamma}\frac{\Delta P}{P(t_i)}$$

i.e.

$$\Delta H(t_i) = -\frac{V(t_{i-1})}{\gamma S}\frac{\Delta P}{P(t_i)} \tag{4}$$

Let call $\Delta H_2(t_i)$ the raising of piston. From (4) we can evaluate $H_2(t_i)$, the height of the piston in each of the n_2 adiabatic steps.

The final height reached is $H_B = H_2(t_N) = 0.344\ cm$.

In the adiabatic compression the temperature of the ideal gas increases at each time-step. The thermal increase at each step can be calculated taking in account the First Law of Thermodynamics for an adiabatic process, i.e. $\Delta U = -P\Delta V$, and the ideal gas property $\Delta U = C_V \Delta T$, where U is the Internal Energy and C_V the molar specific heat at constant volume. We have

$$\Delta T(t_i) = -\frac{P(t_i)}{C_V} S \Delta H(t_i) \tag{5}$$

Figure 3: Height of the piston step by step.$H_0 = 78.2cm$. $H_B = 0.342cm$

The final temperature is $T_B = 283.47K^o$. The entropic change for an adiabatic step is $\Delta\mathcal{S}_U(t_i) \simeq \frac{R}{\gamma}(\frac{\Delta P}{P})^2$ (see Appendix). To reach the initial state P_0V_0, we need $n_3 = 467$ isothermal steps and $n_4 = 143$ adiabatic steps.

Removing n_3 $dw's$ from the piston in isothermal conditions (thermal contact with the heat source T_B) we get the state P_CV_C. For this expansion we have

$$H_3(t_i) = \frac{P_BV_B}{SP(t_i)} - \frac{V_{box}}{S} = H_B + \frac{V_B}{S}(\frac{P_B}{P(t_i)} - 1) \tag{6}$$

from which the raisings of the piston $\Delta H_3(t_i)$ can be evaluated. Observe that in the expansions $\Delta H \geq 0$ and $\frac{\Delta P}{P_0} = -\frac{1}{1033}$. Moreover $\Delta T(t_i) = 0$ and $\Delta\mathcal{S}(t_i) \simeq R(\frac{\Delta P}{P})^2$ (see Appendix)

Finally removing the last n_4 $dw's$ in adiabatic conditions we come back to the initial state V_0, P_0, T_0. In this last process we can evaluate $\Delta H_4(t_i)$ trough a relation similar to (4). $H_4(t_i)$ can consequently be determined. Now at each step the temperature decreases; the entropic change, as before, is $\Delta\mathcal{S}_U(t_i) \simeq \frac{R}{\gamma}(\frac{\Delta P}{P})^2$ (see Appendix)

In Figure 3 we give the height of the piston for each step. Fig 4 reports the raising of the piston for each step.

3 Raising of the single $dw's$

Figure 4: Raising of the piston step by step.Observe the discontinuities in the raising of the piston going from the isothermal process to the adiabatic process and *vice-versa.* i.e. around the states A, B and C

In this section we find the raising of each dw and show that some of them move down (have negative raising) and that the remaining majority move up (have positive raising). The history of the single dw is relevant to evaluate its raising. To this end let us label each dw. The k^{th} dw is the one placed on the piston at the k^{th} time-step ($k \in (1...N)$). So the $N^{th} dw$ is the last one. The raising of the k^{th} dw at the end of the cycle, h_k, is its vertical shift on the support, i.e. the difference between the final and the initial position on the supports.

Really relevant is the order in which they are removed from the piston. Two are the possible ways in which we can start to remove them: a) from the N^{th}, b) from the $(N-L)^{th}$ with $1 \leq L < N$.

In the following we report on both cases for non-random processes: in the case a) we start from the N^{th} and continue with the $(N-1)^{th}$ until the 1^{th}; in the case b) we start from the $(N-L)^{th}$, go on with the $(N-L+1)^{th}$ until the N^{th} and then continue with the $(N-L-1)^{th}$ until the 1^{th}. We call these processes respectively $a-processes$ and $b-processes$. Obviously the expansion process can be done in $N!$ ways. For example you can start with N^{th} dw, go regularly to the $(N-R)^{th}$ then jump to the $(N-2R)^{th}$ and then return to the $(N-R)^{th}$ and continue; otherwise you can start from the $(N-L)^{th}$ go to the N^{th}, jump to the $(N-2L)^{th}$, return to the $(N-L-1)^{th}$ and so on and so on. Once all the possible regular processes have been exhausted one can go to the random-

processes. The nice aspect is that for each process we have a different distribution of the h_k. With the complexity of the distribution of the h_k we recover some of the complexity of the microscopic behavior of the ideal gas.

$a-processes$. Let the first removed be the N^{th}. The individual raising are related to the raising of the piston, therefore it is clear that h_N, the raising of the last dw (which has been on the piston just for one step) is given by $h_N = \Delta H(t_N) = H(t_N) - H(t_{N-1})$ and for the last but one dw (which has been on the piston just for two steps) it is clear that $h_{N-1} = \Delta H(t_{N-1}) + \Delta H(t_N) + \Delta H(t_{N+1}) = H(t_{N+1}) - H(t_{(N-1)-1})$.

Therefore for the $(N-r)^{th}$ dw, with $r \in (0,..,N-1)$, we have

$$
\begin{aligned}
h_{N-r} &= \sum_{i=N-r}^{N+r} \Delta H(t_i) = \sum_{i=N-r}^{N} \Delta H(t_i) + \sum_{i=N+1}^{N+r} \Delta H(t_i) = \\
&= H(t_N) - H(t_{N-r-1}) + H(t_{N+r}) - H(t_{N+1-1}) = \\
&= H(t_{N+r}) - H(t_{N-r-1})
\end{aligned}
\tag{7}
$$

which immediately gives

$$
h_{N-r} = \frac{1}{S}[\,V(t_{N+r}) - V(t_{N-r-1})\,] \tag{8}
$$

This last relation is useful since it enables to appreciate the raising of the dw directly from an inspection of the $P-V$ diagram of the Carnot-cycle : we have only to observe that $V(t_{N+r})$ is the volume occupied by the gas in the *expansion* at pressure $P(t_{N+r})$ and that $V(t_{N-r-1})$ is the volume in the *compression* at the lower pressure $P(t_{N-r-1}) = P(t_{N-r}) - \Delta P$. In this way at each step we can find $\delta V = V(t_{N+r}) - V(t_{N-r-1})$, which is relative to just one ΔP. From the $P-V$ diagram of the Carnot-cycle we see that δV is negative only for the first steps and last steps, respectively. In fig 5, on a schematic representation of the extremities of the cycle, some positive and negative δV's are shown. Therefore we see that the raising h_k is *negative* only for the first and the last steps and is *positive* for all the others. From this analysis, moreover, we can understand that for the $a-processes$ the number of negative raisings depends on how big is ΔP. For $\Delta P \to 0$ that number goes to zero.

The values of h_{N-r} are obtained from equation (8) together with relations (2), (4), (6) and are reported in Figure 6.

Now we go to $b-processes$.

If the first dw removed is the $(N-L)^{th}$ with $1 \leq L < N$, the histories of the single $dw's$ change. The $(N-L)^{th} dw$ has been on the piston during the last $L+1$ steps of the compression and we have

$$
h_{N-L} = \sum_{i=N-L}^{N} \Delta H(t_i) = H(t_N) - H(t_{N-L-1}). \tag{9}
$$

Figure 5: a) Positive and negative values of ΔV around the N^{th} step. b) Positive and negative values of ΔV around the $2N^{th}$ step. The representation is schematic: the step-wise aspect of the cycle is not reported .

For $(N-(L-1))^{th}\, dw$, which has been on the piston for the last L steps of the compression and for the first step of the expansion (that one in which the $(N-L)^{th}$ dw is removed) we have

$$h_{N-(L-1)} = \sum_{i=N-(L-1)}^{N} \Delta H(t_i) + \Delta H(t_{N+1}) = H(t_{N+1}) - H(t_{N-L}) \tag{10}$$

Therefore, if we denote with h^L_{N-r} the $b-processes$ raising, for $r \leq L$, we have

$$h^L_{N-r} = \sum_{i=N-r}^{N} \Delta H(t_i) + \sum_{i=N+1}^{N+L-r} \Delta H(t_i) = H(t_{N+L-r}) - H(t_{N-r-1}) \tag{11}$$

which gives

$$h^L_{N-r} = \frac{1}{S}[V(t_{N+L-r}) - V(t_{N-r-1})] \tag{12}$$

This relation too is useful to appreciate the raising of the $dw's$ directly from an inspection of the $P-V$ diagram of the Carnot-cycle : we can observe that $V(t_{N+L-r})$ is the volume occupied by the gas in the *expansion* at pressure $P(t_{N+L-r}) = P_0(z+N-(L-r))/z$ and that $V(t_{N-r-1})$ is the volume in the *compression* at the pressure $P(t_{N-r-1}) = P_0(z+N-r-1)/z$. So $\delta V = V(t_{N+L-r}) - V(t_{N-r-1})$ is relative to a $\delta P = P_0(2r+1-L)/z$, and it is possible to see that for $\delta P > 0$ we have $\delta V < 0$ and hence $h^L_{N-r} < 0$ and for $\delta P < 0$ we have $\delta V > 0$ and hence $h^L_{N-r} > 0$. For example, for $r = L$ we have $\delta P =$

Figure 6: Distribution of the overall rising for each dw in the $a-processes$. The overall rising is the vertical shift of the single dw on the support i.e. the difference between the final position and the initial position on the support. The top two graphs are magnifications of the initial and final part of the bottom graph and show that the last dw's and the first ones have negative raising

Figure 7: Distribution of the overall rising for each dw in the $b - processes$ for $L = 25$

$\delta P_{\max} = \frac{L+1}{z} P_0 > 0$, to which corresponds the maximum negative raising h^L_{N-L}; for $r = 0$ we have $\delta P = \delta P_{\min} = \frac{1-L}{z} P_0 < 0$, to which corresponds the maximum positive raising h^L_N. The values of h^L_{N-r} can be calculated using relation (12) together with (2), (4), (6).

For $r > L$, the way in which the previous L $dw's$ have been removed has no influence. Therefore, for $r > L$ everything is as in the $a - processes$

$$h^L_{N-r} = h_{N-r} = H(t_{N+r}) - H(t_{N-r-1}) \tag{13}$$

In Fig 7 and Fig 8 we report the distributions of h^L_{N-r} for $L = 25$ and $L = 50$.

We point out that the relationship

$$\sum_{r=0}^{L} h^L_{N-r} = \sum_{r=0}^{L} h_{N-r} \tag{14}$$

is fulfilled for these two processes since the work performed by the ideal gas in the cycle is the same, whichever removing process is performed. This identity can be verified using relations (8) and (12).

We conclude this section pointing out that a deeper insight in the cycle can be obtained through the time-dependent raising $h_{N-r}(t_{N+l})$ for $l \in (1...N)$. The raising we are dealing with in this paper are the $h_{N-r}(t_{2N})$ i.e. the raising of the *$dw's$ at the end* of the cycle.

Figure 8: Distribution of the overall rising for each dw in the $b-processes$ for $L = 50$.

4 Engine work and raising of the $dw's$

The work performed by the ideal gas in our step-wise Carnot-cycle, W, is clearly given by

$$W = \sum_{i=1}^{2N} P_i \Delta V_i \tag{15}$$

For a reversible cycle

$$W_{rev} = \oint PdV \tag{16}$$

It comes from the elementary calculus that it can be written.

$$\oint PdV - \sum_{i=1}^{2N} P_i \Delta V_i \simeq \frac{|\Delta P|}{P}. \tag{17}$$

From physics point of view it is clear that

$$\sum_{i=1}^{2N} P_i \Delta V_i = m\mathbf{g} \sum_{i=1}^{N} h_i \tag{18}$$

where $\mathbf{g}$ is the gravity acceleration; but to prove this relation we need a little of algebra. It is worth while to write down the proof of this relation in order to have an alternative deduction of the relation we have found for h_i.

Observe that using the identity $\sum_{i=1}^{2N} P_0 \Delta V_i = 0$ we can write $\sum_{i=1}^{2N} P_i \Delta V_i$ in the following way

$$\sum_{i=1}^{2N} P_i \Delta V_i = \Delta P \sum_{i=1}^{N} i\ \Delta V(t_i) + \Delta P \sum_{l=1}^{N} (N-l) \Delta V(t_{N+l}) \tag{19}$$

And from $\Delta V(t_j) = V(t_j) - V(t_{j-1})$ and the identity $\sum_{i=0}^{N-1} V(t_i) = \sum_{l=0}^{N-1} V(t_{N-l-1})$ we have

$$\sum_{i=1}^{2N} P_i \Delta V_i = \Delta P \sum_{l=0}^{N-1} \left(V(t_{N+l}) - V(t_{N-l-1}) \right) \tag{20}$$

Now recalling that

$$\Delta P = \frac{1}{z} P_0 = \frac{1}{z} \frac{z\ g}{cm^2} \mathbf{g} = \frac{m}{S} \mathbf{g}$$

we can conclude that

$$\sum_{i=1}^{2N} P_i \Delta V_i = \frac{m}{S} \mathbf{g} \sum_{l=0}^{N-1} [V(t_{N+l}) - V(t_{N-l-1})] = m\mathbf{g} \sum_{l=0}^{N-1} h_{N-l} = m\mathbf{g} \sum_{i=1}^{N} h_i \tag{21}$$

with

$$h_{N-l} = \frac{1}{S} [V(t_{N+l}) - V(t_{N-l-1})] \tag{22}$$

This relation together with the equality (18) can give property (8) in an alternative way.

5 Efficiency of the Engine

The heat quantity Q_a that the engine adsorbs in the n_3 steps performed in thermal contact with the heat reservoir at T_B is given by

$$Q_a = \sum_{i=1}^{n_3} P(t_{N+i}) \Delta V(t_{N+i}) \tag{23}$$

The efficiency of the engine is therefore given by

$$\eta = \frac{\sum_{i=1}^{2N} P_i \Delta V_i}{\sum_{i=1}^{n_3} P(t_{N+i}) \Delta V(t_{N+i})} \tag{24}$$

As is well known, the adiabatic works in the Carnot-cycle cancel each other, therefore

$$\eta = 1 + \frac{\sum_{i=1}^{n_1} P(t_i)\Delta V(t_i)}{\sum_{i=1}^{n_3} P(t_{N+i})\Delta V(t_{N+i})} \tag{25}$$

of course

$$\sum_{i=1}^{n_1} P(t_i)\Delta V(t_i) = Q_0$$

where Q_0 is the heat delivered at the heat reservoir T_0. From relation (29) in the Appendix we know that in the isothermal compression $\frac{-\Delta V}{V(t_i)} = \frac{\Delta P}{P(t_{i-1})}$, therefore

$$Q_0 = \sum_{i=1}^{n_1} P(t_i)V(t_i) \ \left[\frac{\Delta V(t_i)}{V(t_i)}\right] = -RT_0 \sum_{i=1}^{n_1} \frac{\Delta P}{P(t_{i-1})} = -RT_0 \sum_{i=1}^{n_1} \frac{1}{z+i-1} \tag{26}$$

The sum is

$$\sum_{i=1}^{n_1} \frac{1}{z+i-1} = \frac{\partial}{\partial z}\left[\ln(z-1+n_1)! - \ln(z-1)!\right] = \Psi(z-1+n_1) - \Psi(z-1)$$

where $\Psi(z)$ is usually called the 'digamma function', i. e. the logaritmic derivative of the $\Gamma(z)$ function.

For Q_a we can similarly write

$$Q_a = \sum_{i=1}^{n_3} P(t_{N+i})\Delta V(t_{N+i}) = RT_B \sum_{i=1}^{n_3} \frac{1}{z+N-n_3-1+i-1} \tag{27}$$

from which the efficiency is

$$\eta = 1 - \frac{T_0}{T_B} f(z, n_1, n_3, N) \tag{28}$$

where

$$f(z, n_1, n_3, N) = \frac{\Psi(z-1+n_1) - \Psi(z-1)}{\Psi(z-1+N-1) - \Psi(z-1+N-n_3-1)}$$

in our example ($z = 1033, n_1 = 410, n_3 = 467, N = 610$)

$$\eta = 1 - \frac{T_0}{T_B}(1+\epsilon)$$

with $\epsilon = 1.2 \ \ 10^{-3}$. We can therefore conclude that the efficiency of our step-wise ideal engine is smaller that of the corresponding reversible Carnot engine. A result which was expected since the "Dissipated Work" W_D (see Appendix) is positive

$$W_D = \sum_{i=1}^{2N} T(t_i)\Delta \mathcal{S}_U(t_i) > 0$$

Obviously it is also expected that for $\Delta P -> 0$ (i.e. $N -> \infty$) $\eta = 1 - \frac{T_0}{T_B}$

6 Conclusions

The detailed analysis of the classical Carnot cycle we have followed here shows a fruitful complexity in the behaviour of small driving weights, whose energetic gain can stimulate further speculations. A deeper insight in the cycle can be obtained trough the time-dependent raisings $h_{N-r}(t_i)$, for instance it would be interesting to study the effect, on the individual raising $h_{N-r}(t_i)$, of the fact that the adiabatic works in the Carnot-cycle for an ideal gas cancel each other. The estimate of the Efficiency and of the Dissipated Work in our step-wise cycle may be useful to give a deeper insight on the relation among them. In particular we plan to show in a forthcoming paper that the efficiency of an arbitrary non reversible engine running between $T_{\min}$ and $T_{\max}$, $\eta_{Irre}(T_{\min}, T_{\max})$, is equal to the efficiency of a suitable step-wise ideal gas Carnot engine $\eta_{step}(T_{\min}, T_{\max}, \Delta P)$. All these are conceptual aspects. It would moreover be useful to have some practical realization of the actual step-wise Carnot engine since the pattern of the overall raising of the $dw's$ would be preserved in spite of the energy loss due to the friction between the piston and the vessel.

A Entropic changes for Isothermal and Adiabatic Processes of the step-wise Carnot-cycle

1) Isothermal processes

For each time-step t_i we have $P(t_i)V(t_i) = RT$, T being the constant temperature at which the process is performed

To evaluate $\Delta\mathcal{S}_U(t_i) = \Delta\mathcal{S}_{sys}(t_i) + \Delta\mathcal{S}_{env}(t_i)$ we first observe that :

$$\Delta U = 0 \;\Rightarrow \Delta Q = \Delta W = P\Delta V = P(t_i)[V(t_i) - V(t_{i-1})]$$

and

$$\frac{V(t_{i-1}) - V(t_i)}{V(t_i)} = \frac{P(t_i) - P(t_{i-1})}{P(t_{i-1})} \Longrightarrow \frac{-\Delta V}{V} = \frac{\Delta P}{P}. \tag{29}$$

Note that during the expansion $V(t_{i-1}) - V(t_i) > 0$ whereas during the compression $V(t_{i-1}) - V(t_i) < 0$, so in the following we will use $|\Delta V|$, when necessary. So

$$\Delta\mathcal{S}_{sys}(t_i) = \mathcal{S}_{sys}(t_i) - \mathcal{S}_{sys}(t_{i-1}) = \int_{t_{i-1}}^{t_i} \frac{\delta Q}{T} = \int_{t_{i-1}}^{t_i} \frac{PdV}{T} =$$
$$= R\ \ln \frac{V(t_i)}{V(t_{i-1})} = R\ \ln\left(1 - \frac{V(t_{i-1}) - V(t_i)}{V(t_{i-1})}\right) \cong$$
$$\cong R\left[\left(-\frac{V(t_{i-1}) - V(t_i)}{V(t_{i-1})}\right) + \frac{1}{2}\left(\frac{V(t_{i-1}) - V(t_i)}{V(t_{i-1})}\right)^2 + ...\right] \tag{30}$$

and

$$\Delta\mathcal{S}_{env}(t_i) = \mathcal{S}_{env}(t_i) - \mathcal{S}_{env}(t_{i-1}) = \frac{\Delta Q}{T} = R\left(\frac{V(t_{i-1}) - V(t_i)}{V(t_i)}\right) \tag{31}$$

Therefore to the first order, using relations (30) and (31) we have

$$\Delta\mathcal{S}_U(t_i) = R\left[\frac{|\Delta V|^2}{V(t_{i-1})V(t_i)}\right] \cong R\left(\frac{|\Delta V|}{V}\right)^2 \tag{32}$$

and, using (29), the entropic gain in an isothermal step is

$$\Delta\mathcal{S}_U(t_i) \cong R\left(\frac{\Delta P}{P}\right)^2 \tag{33}$$

2) Adiabatic processes

For the adiabatic steps *a priori* we cannot use $P(t_i)V(t_i)^\gamma = C$. Nevertheless for an adiabatic compression-step, from the First Law of thermodynamics we have $\Delta U = -\Delta W = -P\Delta V = P(t_i)[V(t_{i-1}) - V(t_i)] > 0$, but $\Delta U = C_V(T(t_i) - T(t_{i-1}))$, so from the state equation $PV = RT$ we obtain

$$C_V(T(t_i) - T(t_{i-1})) = R[\frac{P(t_i)}{P(t_{i-1})}T(t_{i-1}) - T(t_i)]$$

i.e.

$$(C_V + R)T(t_i) = RT(t_{i-1})\left(\frac{P(t_i)}{P(t_{i-1})} + \frac{C_V}{R}\right) \tag{34}$$

which gives $T(t_i)$ and therefore, since $C_P = C_V + R$

$$V(t_i) = \frac{RP(t_{i-1})V(t_{i-1})}{P(t_i)C_P}\left(\frac{P(t_i)}{P(t_{i-1})} + \frac{C_V}{R}\right)$$

and recalling that $\gamma = c_P/c_V$ this can be written

$$\frac{V(t_i)}{V(t_{i-1})} = \left(1 - \frac{1}{\gamma}\right) + \frac{1}{\gamma}\frac{P(t_{i-1})}{P(t_i)} = \tag{35}$$
$$= 1 - \frac{1}{\gamma}\left(1 - \frac{P(t_{i-1})}{P(t_i)}\right) = 1 - \frac{1}{\gamma}\frac{\Delta P}{P(t_i)}$$

This relation is general for an ideal gas since it connects the final volume to the initial volume for an irreversible adiabatic compression in which the external pressure is suddenly increased.

It follows also that

$$\frac{\Delta V(t_i)}{V(t_{i-1})} = -\frac{1}{\gamma}\frac{\Delta P}{P(t_i)} \qquad i.e. \qquad \frac{\Delta V}{V} \cong -\frac{1}{\gamma}\frac{\Delta P}{P} \tag{36}$$

It is worth-while to observe that relation (35) coincides with the first term of the expansion

$$\frac{V(t_i)}{V(t_{i-1})} = \left(\frac{P(t_{i-1})}{P(t_i)}\right)^{\frac{1}{\gamma}} = \left(1 - \frac{P(t_i) - P(t_{i-1})}{P(t_i)}\right)^{\frac{1}{\gamma}} =$$

$$= 1 - \left(\frac{\Delta P}{P(t_i)}\right)\frac{1}{\gamma} + \frac{1}{2!}\frac{1}{\gamma}\left(\frac{1}{\gamma} - 1\right)\left(\frac{\Delta P}{P(t_i)}\right)^2 +$$

$$+ \frac{1}{3!}\frac{1}{\gamma}\left(\frac{1}{\gamma} - 1\right)\left(\frac{1}{\gamma} - 2\right)\left(\frac{\Delta P}{P(t_i)}\right)^3 + ...$$

so for small ΔP steps we can evaluate ΔV from $P(t_i)V(t_i)^\gamma = C$.

To evaluate $\Delta\mathcal{S}_U$ for an adiabatic step we need only $\Delta\mathcal{S}_{sys}$. From the First Law we have

$$\Delta\mathcal{S}_{sys}(t_i) = \mathcal{S}_{sys}(t_i) - \mathcal{S}_{sys}(t_{i-1}) = \int_{t_{i-1}}^{t_i}\frac{\delta Q}{T} = \int_{t_{i-1}}^{t_i}\frac{C_V dT}{T} + \int_{t_{i-1}}^{t_i}\frac{P dV}{T} = \tag{37}$$

$$= C_V \ \ln\frac{T(t_i)}{T(t_{i-1})} + R \ \ln\frac{V(t_i)}{V(t_{i-1})}$$

which to the first order gives

$$\Delta\mathcal{S}_{sys}(t_i) = C_V\frac{T(t_i) - T(t_{i-1})}{T(t_{i-1})} + R\left(-\frac{V(t_{i-1}) - V(t_i)}{V(t_{i-1})}\right) \tag{38}$$

now, recalling that

$$\Delta T = -\Delta T(t_i) = -\frac{P(t_i)}{C_V}\Delta V(t_i)$$

we have

$$\Delta\mathcal{S}_{sys}(t_i) = -P(t_i)\frac{V(t_i) - V(t_{i-1})}{T(t_{i-1})} + R\left(-\frac{V(t_{i-1}) - V(t_i)}{V(t_{i-1})}\right) =$$

$$= -\frac{P(t_i)}{P(t_{i-1})}R\frac{\Delta V}{V(t_{i-1})} + R\frac{\Delta V}{V(t_{i-1})} = R\frac{\Delta V}{V(t_{i-1})}\left(1 - \frac{P(t_i)}{P(t_{i-1})}\right) =$$

$$= R\frac{\Delta V}{V(t_{i-1})}\left(\frac{\gamma\Delta V}{V(t_{i-1}) + \gamma\Delta V}\right) \cong R\gamma\left(\frac{\Delta V}{V}\right)^2 = \frac{R}{\gamma}\left(\frac{\Delta P}{P}\right)^2 \tag{39}$$

In the adiabatic expansion $\Delta P < 0$ and $\Delta V > 0$, but these changes do not alter the value of the entropic gain we have found in the compression.

From relations (33) and (39) we can conclude that in the step-wise Carnot cycle

$$\oint d\mathcal{S}_U \simeq \frac{\Delta P}{P} \tag{40}$$

Since for each process the number of steps is $\mathcal{N} \cong \frac{P}{\Delta P}$, and for each step $\Delta\mathcal{S}_U(t_i) \sim (\Delta P/P)^2$.

The above estimate of $\Delta\mathcal{S}_U(t_i)$ enables to write down in explicit form the "Dissipated Work" W_D:

$$\begin{aligned} W_D &= \sum_{i=1}^{2N} T(t_i)\Delta\mathcal{S}_U(t_i) = \sum_{i=1}^{n_1} T_0 R\left(\frac{\Delta P}{P(t_i)}\right)^2 + \sum_{i=1}^{n_2} T(t_{n_1+i})\frac{R}{\gamma}\left(\frac{\Delta P}{P(t_{n_1+i})}\right)^2 + \\ &+ \sum_{i=1}^{n_3} T_B R\left(\frac{\Delta P}{P(t_{N+i})}\right)^2 + \sum_{i=1}^{n_4} T(t_{N+n_3+i})\frac{R}{\gamma}\left(\frac{\Delta P}{P(t_{N+n_3+i})}\right). \end{aligned}$$

References

[1] To perform a true reversible process an infinity of steps are needed. Here we consider a finite number of steps. As it is show in Appendix, increasing the steps number we approximate better a reversible cycle.

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[7] It will be available at the following address http://physicsweb.org/TIPTOP/VLAB/

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